EXAMPLE PROOF BY MATHEMATICAL INDUCTION — PROBLEM # 11 of Section 5.2 SOLUTION

<u>TO PROVE</u>: $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for all integers $n \geq 1$.

<u>PROOF</u>: [by Mathematical Induction]

[BASIS STEP]

Let $n = 1$. $\therefore 1^3 + 2^3 + \cdots + n^3 = 1^3 = 1$

$$\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{1(1+1)}{2}\right]^2 = 1, \text{ by substitution.}$$

$$\therefore 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \text{ for } n = 1.$$

[END OF BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 1$.

Suppose $1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$. [The INDUCTIVE HYPOTHESIS]

[We N.T.S. $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2$.]

$$\therefore (1^3 + 2^3 + \cdots + k^3) + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

by the Inductive Hypothesis and Substitution,

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4}$$

$$\therefore (1^3+2^3+\cdots+k^3)+(k+1)^3 = \frac{(k^2+4(k+1))(k+1)^2}{4}$$

$$= \frac{(k^2+4k+4)(k+1)^2}{4}$$

$$= \frac{(k+2)^2(k+1)^2}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$\therefore 1^3+2^3+\cdots+(k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2 = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2.$$

$\therefore$ By Direct Proof, For all integers $k \geq 1$,

if $1^3+2^3+\cdots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$,

then $1^3+2^3+\cdots+(k+1)^3 = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2$.

[END-of-Inductive Step]

$\therefore$ For all integers $n \geq 1$, $1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$,

by the Principle of Mathematical Induction.

QED.